"A Radial Area-Scale." By R. W. K. Edwards, M.A. Communicated by Professor A. G. Greenhill, F.R.S. Received January 28,—Read March 3, 1904.

This is a contrivance for finding the area of a plane figure by means of a transparency. The design in the transparency consists of a number of radiating lines. Each of these lines is graduated.

There are various patterns of this design, and their respective claims to convenience and accuracy form a wide field for discussion. In the accompanying transparency (reproduced in the figure), which is fairly simple and effective, there are eleven straight lines radiating from a point at equal angles. The way in which the transparency is used is as follows :—

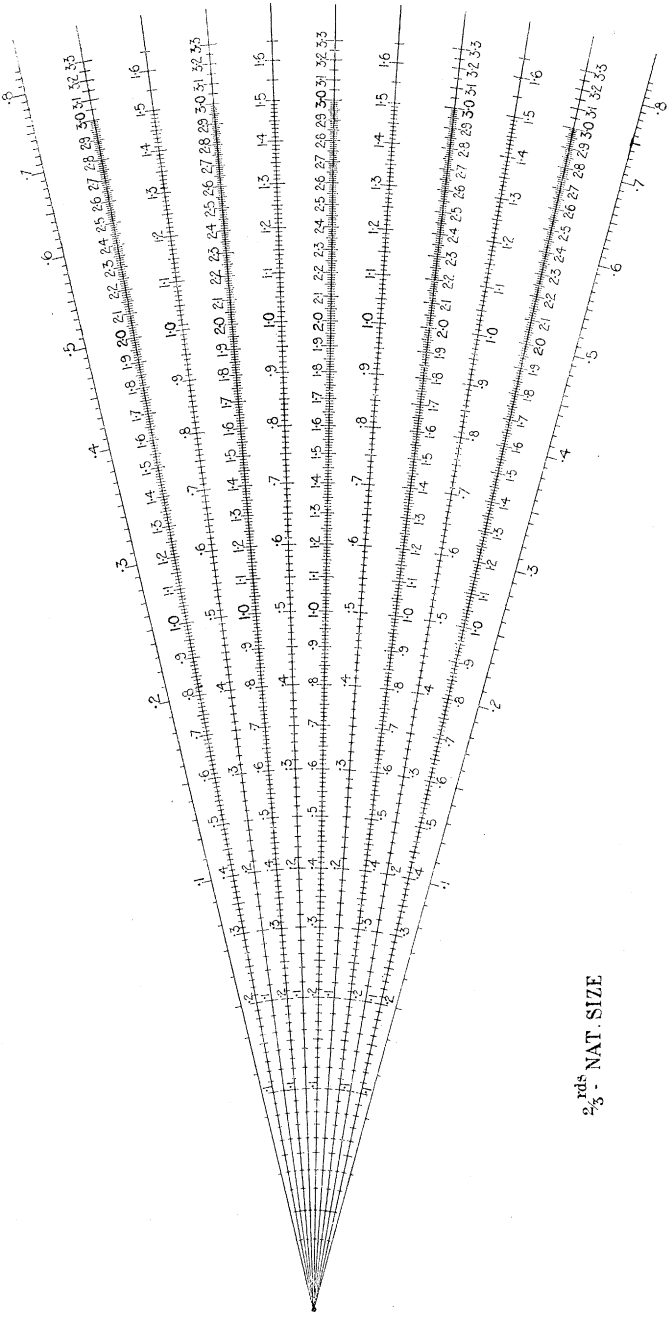
The figure whose area is to be found is placed under the transparency, in close contact with it, so that its contour lies *just* between the two outside lines of the transparency, *i.e.*, so that each outside line touches the contour, or passes through a cusp or angular point, or contains some rectilineal portion of the contour. Each of the radiating lines thus becomes a tangent or transversal, or contains a side of the figure. The graduations of the right-hand points of intersection of these transversals are read and added together; then the graduations of the left-hand points of intersection are read and added together. The second sum is subtracted from the first; and the result records the number of square inches in the figure.

It will be seen that if each of the outside lines touches, or passes through an angular point of the figure, there will be eighteen graduations to be read—those on the outside lines cancelling each other. If one of the outside lines contains a rectilineal portion of the contour, there will be twenty graduations to read; if both outside lines do so, twenty-two graduations must be recorded.

If a quicker use of the area-scale, with less chance of accuracy, is desired, the figure to be quadrated may be placed so as to lie just between the first and ninth, or the first and seventh, or the first and fifth lines, in which cases fourteen, or ten, or six graduations will be read respectively.

If the figure is too large to be included between the outside lines, it may be divided into two parts by a straight line drawn across it, or into three parts by a pair of straight lines inclined to one another at the same angle as the outside lines, which may be done by means of the cardboard slip accompanying the diagram. The second of these methods of dividing up the figure may be also employed when it is desired to obviate the inaccuracy that may result from the two outside lines otherwise being tangent to the figure.

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Principle of the Area-Scale.

Suppose the equation of a curve to be

$$r^2 = a + b\theta + c\theta^2 + d\theta^3,$$

the area between the curve, the initial line, and the radius vector making angle 2θ with the initial line, is $\frac{1}{2}\int_0^{2\theta} r^2\, d\theta$; *i.e.*,

$$a\theta + 2b\theta^2 + \tfrac{8}{3}c\theta^3 + 4d\theta^4, \text{ } i\,e., \text{ S, say.}$$

Now if r_0, r_2 be the outside radii of the sector so quadrated, and r_1 be the radius vector bisecting the angle between them, it can be seen that

$$\begin{aligned} r_0^2 &= a, \\ r_1^2 &= a + b\theta + c\theta^2 + d\theta^3, \\ r_2^2 &= a + 2b\theta + 4c\theta^2 + 8d\theta^3; \end{aligned}$$

whence, assuming $(Ar_0^2 + Br_1^2 + Cr_2^2)\ \theta = S$, and solving three of the four simultaneous equations for A, B, C, we get values $A = \frac{1}{6}$, $B = \frac{2}{3}$, $C = \frac{1}{6}$, which also satisfy the fourth.

We have then

$$S = \tfrac{1}{6}\theta\ (r_0^2 + 4r_1^2 + r_2^2).$$

The design on this particular transparency is made on the assumption that each of the ten separate portions of the curve between the first and third, the third and fifth, . . . the ninth and eleventh radiating lines, approximates to some member of the above family of curves.

Anyone acquainted with the discussion of areas, in Cartesian co-ordinates, in terms of a series of equidistant ordinates, and their common distance apart, given in Bertrand's *Calcul Intégral*, Section 363, will see an analogy in the foregoing expression of the area of a sector in terms of the squares of three equally inclined radii and their common inclination. Other designs of the transparency can be made on the assumption of larger portions of the curve approximating to members of families of curves in which r^2 is equal to a rational algebraical function of θ in ascending powers of θ, of degree higher than the third. On the whole it is easier, and possibly more effective, to use the method employed above, in the same way as "Simpson's Rule" uses the corresponding theorem in Cartesian co-ordinates. There is no reason to suppose that "Simpson's Rule" gives a less accurate approximation in the generality of cases than "Weddle's Rule" * or the numerous rules deducible in terms of the co-efficients calculated by Cotes.

An investigation of the family of curves

$$r^2 = a + b\theta + c\theta^2 + d\theta^3$$

* See Boole's 'Finite Differences.'

shows that they form a large variety of spirals. An important point to notice is that if, as usual, ϕ denotes the inclination of a radius to the tangent at its extremity, we have tan ϕ, *i.e.*, $rd\theta/dr$, equal to

$$\frac{2\ (a+b\theta+c\theta^2+d\theta^3)}{b+2c\theta+3d\theta^2},$$

which, in general, cannot be zero unless at the origin, when the curve passes through the origin. This means that error is likely to be perceptible when the curve quadrated is such that any tangent to it passes through the point of radiation; *e.g.*, when the outside lines are tangents to the curve; or when the curve is re-entrant in such a way that any one of the radiating lines gives three or four readings. The design may be expected to give the best results, therefore, when arranged so that the outside lines past through cusps or angular points of the curve, and so that no radiating line crosses it more than twice. For curves in which there are no sharp points, it is best, therefore, to divide into two or three areas.

There is no reason why the presence of points of contrary flexure should be supposed to vitiate the results. For the equation for θ giving the positions at which such points occur in the stated family of curves, will be found to be, in general, of the sixth degree.

Calculable deviations from strict accuracy may be expected in the case of nearly all the well-known regular curves quadrated by means of the area-scale. If, as in the present pattern, the angle between the outside lines is half a radian, these deviations will be found to be insignificant, except in the cases of oval curves touching the outside lines and not treated as suggested on p. 292.